Appears in *Fractal Frontiers*. Eds. M. M. Novak and T. G. Dewey, World Scientific (Singapore), 1997, pp. 63–72.

LÉVY KINETICS IN SLAB GEOMETRY: SCALING OF TRANSMISSION PROBABILITY

A. DAVIS AND A. MARSHAK
NASA's Goddard Space Flight Center, Climate and Radiation Branch, Greenbelt, MD 20771, USA

We revisit a classical problem in kinetic theory, transport through a slab of thickness L, when particle free-path distributions have infinite variance but finite moments of order $q < \alpha < 2$. After many (isotropic) scatterings, particle trajectories $\mathbf{x}(n)$, $n \geq 0$, are essentially 3-dimensional discrete-time Lévy-stable random walks ("Lévy-flights") of index α truncated at random times when a boundary is crossed. Starting at $\mathbf{x}(0) = \mathbf{0}$ with a step into the $z > 0$ half-space, trajectories end in either transmission ($z(n_T) \geq L$, $n_T \geq 0$) or reflection ($z(n_R) \leq 0$, $n_R \geq 1$). The classic case is for exponentially distributed steps with mean-free-path ℓ, leading to $\langle n_T \rangle_e \sim (L/\ell)^2$ and normalized flux (transmission probability) $T_e(L) \sim (L/\ell)^{-1}$. Numerical simulations of Lévy/Gauss transport yield $T_\alpha(L) \propto L^{-\alpha/2}$ whether $\ell < \infty$ ($1 < \alpha \leq 2$) or $\ell = \infty$ ($0 < \alpha \leq 1$). To derive this result from the standard scaling relation, $\langle z(n)^\alpha \rangle \propto n$ (recast here as $\langle n_T \rangle_\alpha \propto L^\alpha$), zero-crossing events for the discrete-time Lévy-stable process must be described by the correlation dimension $D_2 = 1/2$ (independent of α), rather than the better-known Hausdorff dimension $D_f = \max\{1-(1/\alpha), 0\}$. Applications to Earth's climatic equilibrium and cloud remote sensing are discussed (including the effect of non-isotropic scattering described by kernels with forward peaks).

1 Introduction and Overview

Their property of self-similarity and their probability density functions (PDFs) with power-law tails have ensured Lévy flights a prominent place in the stochastic modeling of dynamical, chemical, and even biological systems.[1] Among all applications of Lévy flight theory, the closest to their definition as random walks are in the realm of kinetic transport, exploiting directly the defining property of Lévy deviates (namely, stability under addition). In this specific context, previous studies using Lévy-stable step distributions have considered unbounded[2,3] and semi-infinite[4,5] domains. Slab geometry is the next logical step and it is the appropriate one for modeling a wide variety of physical systems ranging from shields in nuclear reactors to planetary atmospheres.

To date, the problem of transport through plane-parallel scattering media has been treated in full detail only for exponential and Gaussian particle free-path distributions, respectively in mathematical frameworks provided radiative transfer[6] and diffusion[7] theories. By focusing on the most basic scaling property (particle transmission probability with respect to slab thickness), we further the theory of particulate transport in finite media under circumstances where free-path distributions have power-law tails.

In the next section, required elements of random-walk theory are surveyed, including a recent result by Frisch and Frisch[5] on the universality of the probability of escape from a half-space. In section 3, we obtain numerically and as corollaries of the Frisch's theorem scaling relations for steady-state transmission probability and the associated mean transit time for Gaussian and symmetric Lévy step-distributions. We summarize our findings in section 4 and discuss in section 5 some of their more important implications for current issues in atmospheric radiation. In an appendix, we show how diffusion —normal or not— can be adapted to situations (such as the interaction of clouds with solar photons) where the scattering kernel is peaked in the forward direction.

2 Background and Notation

2.1 Unbounded Random Walks

Consider a d-dimensional discrete-time random walk starting at the origin ($x(0) = \mathbf{0}$). After n identically distributed steps $\{s(j) \in \Re^d, j = 0,\ldots,n-1\}$ and $n-1$ scatterings, the random position-vector of the particle is

$$x(n) = \sum_{j=0}^{n-1} s(j) \in \Re^d, \; n > 0. \tag{1}$$

We use

$$s = \|s\|_2 = \sqrt{s^2} \tag{2a}$$

to denote the random (Euclidean) distance covered between two scattering events, and

$$\Omega = s \,/\, \|s\|_2 \in \Xi_d = \{x \in \Re^d, \|x\|_2 = 1\} \tag{2b}$$

for the random unit-vector indicating direction of travel, assumed independent of s. Thus

$$s_m = s\Omega_m \; (m = 1,\ldots,d) \tag{3}$$

denotes a component of the random step-vector. In some cases, a common PDF for all the s_m's is used; in others, a law for s is prescribed and Ω is drawn from the uniform distribution on Ξ_d.[a]

The most-studied type of random walk by far uses (zero-mean) Gaussian steps, in which case:

$$\langle x(n)^2\rangle_G = [d\sigma^2]n, \tag{4}$$

where $\sigma = \sqrt{\langle s_m{}^2\rangle}$ ($m = 1,\ldots,d$) is the root-mean-square (rms) step-size in each direction. Constant (degenerate) steps are also very popular, especially when a discrete Ω-space keeps the particles on a regular grid. Exponential step-distributions with mean-free-path (mfp) $\ell = \langle s\rangle$ play a key role in kinetic theory (cf. Appendix). In both cases, the same scaling property is obtained as in (4) with $\sigma^2 = \ell^2/d$, replacing however "=" by "≈", in the limit $n \gg 1$ (law of large numbers, applicable here since step variance $\sigma^2 < \infty$). In all of the above situations, particle trajectories are described as Brownian motion (Bm).

There was been a sustained interest[1] in the physics community in random walks with (symmetric) Lévy-stable step distributions, better described[8] as Lévy "flights" because of the rare but large jumps that lead to $\langle s^2\rangle = \infty$, and possibly even $\langle s\rangle = \infty$. The stability property of symmetric Lévy deviates here reads as

$$\sum_{j=0}^{n-1} s_m(j) \overset{d}{=} n^{1/\alpha} s_m(0) \; (m = 1,\ldots,d) \tag{5a}$$

where "$\overset{d}{=}$" means "identical in distribution." This is equivalent[9] to

$$\langle \exp[iks_m]\rangle_L = \exp[-(c|k|)^\alpha] \; (m = 1,\ldots,d), \tag{5b}$$

[a] In the Appendix, we show that this isotropy assumption is not too restrictive as long as the system is dominated by high orders-of-scattering. We rescale the steps s and the orders-of-scatterings n to accommodate situations where the scattering probability peaks in the forward direction, thus inducing directional correlations in the otherwise random walk.

where $0 < \alpha < 2$ is the Lévy "index" (moments $\langle |s_m|^q \rangle$ of order $q \geq \alpha$ are divergent) and $c \geq 0$ is an amplitude parameter, like σ in the Gaussian case (which is retrieved in the limit $\alpha \to 2^-$). Either of Eqs. (5a,b) uniquely defines a symmetric Lévy law but its PDF is known in closed-form only if $\alpha = 1$: $dP(s_m) = \pi^{-1}(ds_m/c)/[1+(s_m/c)^2]$ (Cauchy's law).

Combined with the definition in Eq. (1), (5a) leads to a counterpart of the scaling relation in Eq. (4) for Lévy flights:

$$\langle x(n)^\alpha \rangle_L \sim c^\alpha n \quad (n \gg 1) \tag{6}$$

for a finite sample, the prefactor increasing logarithmically with sample size.

2.2 Semi-Infinite Geometry, 1 – The Universality of Escape Probability

Consider random walks confined to the half-space $x_d \geq 0$; for instance, in $d = 3$ we have $x_1 = x \in \Re$, $x_2 = y \in \Re$, and $x_3 = z \in \Re^+$. Trajectories start at $x_d = 0$ with a step where $s_d(0) > 0$ and terminate at the first occurrence of $x_d(n) = \sum_0^n s_d(i) < 0$, thus defining the random escape time n_R. Frisch and Frisch recently showed[5] that, under quite general conditions,

$$\text{Prob}_\infty\{n_R > N\} = \sum_{N+1}^{\infty} p_\infty(n_R) = \frac{(2N)!}{2^N (N!)^2}, \; N \geq 0. \tag{7a}$$

Their result is universal with respect to the distribution for s_d. For $N = 0, 1, 2, 3, \ldots$: $\text{Prob}_\infty\{n_R > N\} = 1, 1/2, 3/8, 5/16, \ldots$; Sterling's formula yields asymptotically

$$\text{Prob}_\infty\{n_R > N\} \sim \frac{1}{\sqrt{\pi N}}, \; N \gg 1. \tag{7b}$$

Thus $p_\infty(n_R) \sim n_R^{-3/2}$, hence $\langle n_R^q \rangle = \infty$ if $q \geq 1/2$; in particular, the mean escape time is infinite.

The special case where path distributions are exponential is relevant to (monochromatic) radiative transfer theory which is based on a Boltzmann-type equation for photons;[6] in such radiation problems, scattering probability is denoted "λ" or "ϖ_0" and there is often a small probability of absorption $1-\lambda = \varepsilon$ at each interaction. Bulk absorptance $A(\varepsilon)$ goes as[7] $\sqrt{\varepsilon}$; identifying $A(\varepsilon)$ with a truncation of the asymptotic distribution function in (7b), $A(\varepsilon) = \text{Prob}_\infty\{n_R > n^*(\varepsilon)\}$, leads to $n^*(\varepsilon) \approx 1/\varepsilon \gg 1$. As a result of the remarkable universality of (7b), "$\sqrt{\varepsilon}$-type" laws prevail in non-LTE,[2] spectral line,[3–5] and broad-band[10] transfer problems where non-exponential effective free-path distributions arise.

2.2 Semi-Infinite Geometry, 2 – Interpretation in Terms of Fractal Geometry

In the above, we implicitly focused on events that define the "zero-crossing" set of the random process $x_d(t_n) = x_d(n) \in \Re$, $t_n = n\delta t \geq 0$, in the continuous time limit ($\delta t \to 0$, $n \to \infty$). In particular, Equation (7b) reflects the well-known fact that the zero-set of Bm ($\alpha = 2$ steps) has Hausdorff dimension $D_f = 1/2$. The heuristic argument leading to this uses (a) the graph dimension $D_g = 3/2$ of Bm (which is self-affine with Hurst exponent $H_1 = 1/2 = 2-D_g$) and (b) the fractal intersection theorem: $D_f = \max\{D_g-1,0\}$. The same reasoning tells us that the zero-set of Lévy flights of index α has fractal dimension $D_f(\alpha) = \max\{D_g(\alpha)-1, 0\} = \max\{1-(1/\alpha), 0\}$ since $H_1(\alpha) = \min\{1/\alpha, 1\}$ follows from Eqs. (5b–6).[11] However, the result in Eq. (7b) is independent of $\alpha \leq 2$.

This paradox is resolved by noting that the scaling in (7b) is not directly related to the Hausdorff/box-counting dimension used implicitly in the *geometrical* argument based on the intersection theorem. Rather (7b) is a (2-point) *statistical* statement implying that the correlation dimension D_2 of the zero-crossing set is 1/2 for all α's. Indeed, the question of interest is not 'Does a time-interval of length τ positioned at random contains any number of zero-crossings?' but rather 'Given a zero-crossing at t_0, is there exactly one other such event at a range $\leq \tau = n_R \delta t$?'. Introducing codimensions, the probabilistic answers are: Prob{'yes'} $\propto \tau^{1-D_f}$, as $\tau \to 0$ (with δt), to the former question; and Prob{'yes'} $\propto \tau^{1-D_2}$ for the latter. Note that the second question only makes sense for a large but *finite* number of points.[12]

The key assumption in the Frisch's derivation is the statistical independence of the sequence of steps in the random walk. For Gaussian ($\alpha = 2$) or Lévy ($0 < \alpha < 2$) steps, $x_d(t)$ is a strictly scale-invariant process with stationary increments, hence

$$\langle [x_d(t+\tau)-x_d(t)]^q \rangle \sim \tau^{qH(q)}, \tag{8a}$$

where

$$\begin{cases} \text{Gauss: } H(q) \equiv 1/2 \\ \text{Lévy: } H(q) = \min\{1/\alpha, 1/q\} \end{cases} \tag{8b}$$

In the latter ($\alpha < 2$) case, the moments of order $q \geq \alpha$ are actually infinite (using *bone fide* ensemble-averaging); the scaling in (8a,b) applies only to *finite* samples.[13]

Setting $q = 2$ in Eq. (8a) for τ and for 2τ, we obtain[8]

$$\langle [x_d(t+2\tau)-x_d(t+\tau)][x_d(t+\tau)-x_d(t)] \rangle = (2^{2H(2)-1} - 1) \times \langle [x_d(t+\tau)-x_d(t)]^2 \rangle. \tag{9}$$

The Frisch's requirement is therefore $H(2) = 1/2$ and is, according to Eq. (8b), verified by all types of Lévy flight (of finite length).

It is noteworthy that $H(2)$ in Eq. (9) is related to the Hurst exponent $H_1 = H(1)$ only by inequality: namely, $H(1) \geq H(2)$.[14] A famous example of the limiting (equality) case is fractional Bm (fBm) which has $H(q) \equiv H_1 \in (0,1)-\{1/2\}$, hence the zero-set capacity dimension $D_f(H_1) = \max\{D_g(H_1)-1, 0\} = 1-H_1 \neq 1/2$. So, if $x_d(n)$ is described by fBm, it does not belong to the universality class described in the previous section.[b]

3 Transmission Through a Finite Plane-Parallel Medium

We now confine the particle's random walk between the hyper-planes $x_d = 0$ (where it starts) and $x_d = L < \infty$ (where it ends, in the event of a transmittance).

3.1 Gaussian Case

We now ask what the mean transit time n_T is for a transmitted particle. To answer this question, we "solve" Eq. (4) for n when $x_1(n) \approx \ldots \approx x_d(n) \approx L$, hence $\langle x(n)^2 \rangle_G \approx dL^2$:

$$\langle n_T \rangle_G = \sum_{n_T=0}^{\infty} n_T p_L(n_T) \sim (L/\sigma)^2, \; \sigma \ll L < \infty. \tag{10}$$

[b] Solar photons scattering off cloud droplets for instance have positive directional correlations, hence "persistence" in the same sense as used in Ref. 8, i.e., $H(2) > 1/2$ in Eq. (9); however, these correlations decay in finite time (see Appendix for details). So, rather than changing the overall scaling to one with $H(2) > 1/2$, they introduce an inner time-scale (corresponding to a few forward scatterings) that is naturally accounted for in the diffusion approximation by defining the "transport" mfp.

The implicit statistical condition for this approximate method of "solution" to work is that the PDF of $x(n)$ is reasonably narrow for large n; this is certainly true for Gaussian processes in the following sense: $\langle x(n)^2\rangle_G > \langle \|x(n)\|_2\rangle_G{}^2$ (Schwartz's inequality) but by a constant factor of $O(1)$, for given d.

The domain boundary at $x_d = L$ acts physically as a Brownian particle absorber, so we can estimate the probability of reaching it simply by truncating the ($L = \infty$) distribution in Eq. (7b):

$$T_G(L) \approx \mathrm{Prob}_\infty\{n_R \gtrsim \langle n_T\rangle_G\} \sim (L^2/\sigma^2)^{-1/2} \sim (L/\sigma)^{-1},\ L \gg \sigma. \tag{11}$$

Exponential path distributions in $d = 3$ (mfp = ℓ) have been extensively studied in the framework of linear transport theory (primarily for photons and neutrons); in this case, the "asymptotic" or "diffusion" solution,[15] valid in the limit L/ℓ = optical thickness $\gg$ 1, is $T_e(L) \approx 1/[1+constant\times(L/\ell)]$ where the constant depends on details of the scattering and illumination. Since $T_e(L) \sim (L/\ell)^{-1}$, this case is in the universality class defined by Gaussian steps (which have $\ell = \langle \|s\|_2\rangle_G \approx \sqrt{\langle s^2\rangle_G} = \sigma\sqrt{d}$).

3.2 *Lévy Case*

The mean transit time for a bounded Lévy flight through a slab of finite thickness L can be estimated from the (finite-sample) scaling property in Eq. (6) in the same way as in the (ensemble-average) Gaussian case and Eq. (4):

$$\langle n_T\rangle_L \sim L^\alpha/c^\alpha,\ c \ll L < \infty. \tag{12}$$

However, the interpretation is somewhat different because Lévy processes are far from being narrowly distributed. The statistical moment of order $q = \alpha$ in Eq. (6) is logarithmically divergent with sample size, meaning that estimates are dominated by the largest event. The interpretation of the scaling relation in Eq. (12) is that, on average, a deviation of magnitude L will almost surely occur before or soon after n_T steps are taken.

We can now invoke the universality of the asymptotic escape probability distribution in (7b) for $L = \infty$ to obtain the transmission probability from (12):

$$T_\alpha(L) \approx \mathrm{Prob}_\infty\{n_R \gtrsim \langle n_T\rangle_L\} \sim (L^\alpha/c^\alpha)^{-1/2} \sim (L/c)^{-\alpha/2},\ L \gg c. \tag{13}$$

This relation is readily verified numerically in $d = 1$ with the following pseudo-code that uses a Box/Muller-type algorithm for generating symmetric Lévy-stable deviates:[16]

```
set:  constant alpha in (0,2]
set:  constant c = 1
set:  constant L > 0
set:  constant N_tot >> 1
set:  variable N_L = 0
set:  variable N   = 0

 while N < N_tot, do:
1  z = 0
   s = c*Sym_Levy_1(alpha)
   if N = 0, then: s <- abs(s)
   N <- N + 1
   z <- z + s
   if z < 0, then: go to 1
   if z > L, then: N_L <- N_L + 1, go to 1

 output:  log(L), log(N_L/N_tot)
```

```
function:  Sym_Levy_1(alpha)

Generates unitary symmetric Lévy deviate.
Uses a pseudo-random number generator rnd(),
uniform on (0,1), and constant pi = 3.14...

  a_inv     = 1.0/alpha
  a_inv_m_1 = a_inv-1.0

  exp_rv = -log(rnd())
  phi    = pi*(0.5-rnd())
  a_phi  = alpha*phi

  Sym_Levy_1 = ( sin(a_phi)/cos(phi)**a_inv )
         *( cos(phi-a_phi)/exp_rv )**a_inv_m_1
```

4 Summary

We have investigated analytically and numerically the scaling properties of particle transport by normal (Gauss) and anomalous (Lévy) diffusive motion through a slab of finite thickness L. In the process, we have argued that the operationally well-defined correlation (D_2) and box-counting (D_f) dimensions of the zero-crossing sets for Lévy flights of large but finite length differ: $D_f = \max\{1-(1/\alpha), 0\}$, and $D_2 = 1/2$, $0 < \alpha < 2$. We have also shown (in the Appendix) how to bring random walks with directional correlations into the fold of diffusion theory by appropriate spatio-temporal rescaling.

5 Application: How Cloud Inhomogeneity Reduces the Earth's Albedo

Lévy (and otherwise "stretched") photon free-path distributions have been used in radiative transfer for a number of primarily astrophysical applications.[2–4] The main motivation for using them (in lieu of Brownian motion) here, in atmospheric radiation modeling, is to emulate the overall ("mean-field") effect of the spatial variability of the density of scatterers —primarily cloud droplets— which is still very poorly understood.

In stratus cloud decks (which look rather bland from below, above, and inside), liquid water density is observed[17] to vary by more than an order of magnitude over scales ranging from ≈10 m to ≈10 km, furthermore obeying multifractal 2-point statistics.[18] Other cloud types exhibit even stronger variability. In-cloud extinction (the inverse of *local* photon mfp) is proportional to density; so photon mfp —much like "visibility" through an airplane window— varies inversely with density,[c] from only a few meters to several hundred. Recent analyses of liquid water measurements at sub-mfp (cm to m) scales have uncovered considerable variability.[19] These rapid density fluctuations are guaranteed to perturb the photon free-path distribution away from the standard exponential case and this perturbation is necessarily in the direction of statistically longer paths.[20,21] In absence of cloud, incoming solar photons have only about a 1÷20 chance of being (Raleigh) scattered by the molecular atmosphere, implying a mfp in excess of 100 km at an altitude of ≈8 km where density is average (≈e times less at sea level). The atmosphere's aerosol load (dust, soot, volcanic, organic, etc.) is highly variable in space and time; in all cases, it reduces photon mfp, especially in horizontal directions close to the Earth's surface. In summary, photons can be "trapped" in optically dense regions (i.e., inside clouds), traveling less than a meter between scatterings, and they can "fly" many kilometers from space-to-surface or vice-versa, cloud-to-cloud, cloud-to/from-surface, or cloud-to/from-space. Power-law (or Lévy-stable) distributions are therefore well-suited to model photon free-paths globally.

In Appendix A, we recall that the various photon mfp values quoted above must be converted into

$$\text{"transport" mfp} = \text{(usual) mfp} / (1-g) \tag{14}$$

before being identified with ℓ in sections 2–3; numerically, $g \approx 0.75$–0.85 in typical aerosol layers and clouds, so $(1-g)^{-1} \approx 4$–7, and $g = 0$ for molecular Raleigh scattering. In the following discussion for $d = 3$, we limit ourselves to $1 < \alpha \le 2$, so that we can

[c] This connection is non-trivial for *true* mfp (an inherently non-local quantity), as opposed to *local* mfp (the inverse of local extinction).[20]

take the (transport) mfp $\ell = \langle \|s\|_2 \rangle_L$ as finite. The new result in Eq. (13) can by merged with the standard one in Eq. (11) to read as

$$T_\alpha(L) \sim \left(\frac{L}{\ell}\right)^{-\alpha/2}. \tag{15}$$

So, at given total optical depth (L/ℓ, that we assume ≈1 or larger), the more inhomogeneous the cloudy atmospheric column (Lévy index α decreases), the more transmissive, hence less reflective, it becomes. This trend has been observed in many numerical studies of the bulk radiative properties of heterogeneous cloud models fractal or not.[22] This means that using the standard ($\alpha = 2$) model introduces a systematic bias that is now being corrected for in the radiative modules of climate forecasting models,[23] and soon in cloud remote-sensing schemes based on passive radiometry.[24]

Acknowledgments

This work was supported by the Environmental Sciences Division of U.S. Department of Energy (grant DE-A105-90ER61069 to *NASA's Goddard Space Flight Center*) as part of the *Atmospheric Radiation Measurement* (ARM) program. Drs. T. Bell, R. Cahalan, S. Lovejoy, M. Shlesinger, and W. Wiscombe are thanked for helpful discussions.

Appendix: *Diffusion Theory for Directionally-Correlated Random Walks*

In many situations of physical interest, differential cross-section for scattering is not isotropic; however, it often depends only on $\Omega(i)\cdot\Omega(i+1) = \cos\theta_s$, the cosine of the scattering angle, not on azimuthal angle(s). In this case, the so-called "asymmetry factor"

$$g = \langle \Omega(i)\cdot\Omega(i+1)\rangle = \langle \cos\theta_s\rangle \in [-1,+1] \tag{A.1}$$

is a first-order descriptor of the positively ($g > 0$) or negatively ($g < 0$) weighted PDF of $\cos\theta_s$. Extreme examples: if $g = 1$, there is no scattering *per se* because it is always in the forward direction; if scattering is isotropic, then $g = 0$, and Ω in Eq. (2b) is uniformly distributed on Ξ_d. Isotropy is a reasonable approximation for neutron scattering in a moderator, not for solar photons in clouds where[25] $g \approx 0.85$.

In this appendix, we show that the effect of positive ($g > 0$) directional correlations —in essence, an enhancement of ballistic travel— can be absorbed into a rescaling of the free paths by $(1-g)^{-1} > 1$, and the number of scatterings by $(1-g) < 1$. With these precautions, we can continue to use diffusion —either normal (Gauss-type steps) or anomalous (Lévy-type steps)— to model particle transport.

A.1 Forward-Peaked Scattering Kernels for $d = 1, 2, 3$

It is convenient, especially for numerical simulations, to have a simple model for the scattering kernel (normalized differential cross-section, or scattering-angle PDF) that, beyond $\theta_s = \cos^{-1}[\Omega(i)\cdot\Omega(i+1)]$, depends on a few adjustable parameters. In $d = 3$ radiative transfer, the kernel is called a "phase-function" and the Heney–Greenstein model[26]

$$p_3(\theta_s) = \frac{1}{\sigma}\times\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi}\right)\frac{1-g^2}{[1+g^2-2g\cos\theta_s]^{3/2}},\ d\Omega = \sin\theta_s d\theta_s d\varphi, \tag{A.2}$$

is by far the most popular. This is partially due to the existence of an inverse cumulative probability $\cos\theta_s = P^{-1}(\xi)$, $0 \le \xi \le 1$, where $P(X) = \text{Prob}\{-1 \le \cos\theta_s \le X\}$ in closed-form for the purposes of Monte Carlo simulation, partially due to its simple, azimuthally independent, spherical-harmonic expansion on Ξ_3: $<p_3(\theta)|Y_l(\theta)> = g^l$, $l \ge 0$.

A counterpart for this model was developed[27] specifically for simulations in $d = 2$:

$$p_2(\theta_s) = \frac{1}{\sigma}\times\frac{d\sigma}{d\Omega} = \left(\frac{1}{2\pi}\right)\frac{1-g^2}{1+g^2-2g\cos\theta_s}, \; d\Omega = d\theta_s. \tag{A.3}$$

Here too, $P^{-1}(\xi)$ is expressible in closed-form, and the cosine-Fourier decomposition of $p_2(\theta)$ on Ξ_2 is simple: $<p_2(\theta)|\cos j\theta> = g^j$, $j \ge 0$.

Sometimes (e.g., in §3.2), it is sufficient to conduct transport simulations in $d = 1$ where direction-space reduces to $\Xi_1 = \{\pm 1\} = \{\cos\theta_s;\ \theta_s = 0,\pi\}$; here scattering is either forward, with probability $p_1(\theta = 0) = (1-g)/2$, or backward, with $p_1(\theta = \pi) = (1+g)/2$.

A.2 Multiple Scattering as a Random Walk in Direction-Space

Let $\Omega(0)$ be the initial direction of propagation in (or position on) Ξ_d; by symmetry, the average position $\langle\Omega(n)\rangle = \langle(\Omega_1(n),\ldots,\Omega_d(n))\rangle = \Omega(0)$ for any number of scatterings n. By taking $\Omega(0)$ as the polar axis, we can use $\theta(n) = \cos^{-1}(\Omega(0)\cdot\Omega(n))$ to measure the (great-circle) distance on Ξ_d between departure and arrival points; so we will have $\langle\theta(n)\rangle = 0$, by symmetry. From (A.1), we know that $\langle\Omega(0)\cdot\Omega(1)\rangle = \ldots = \langle\Omega(n-1)\cdot\Omega(n)\rangle = g$. We can show by induction that

$$\langle\cos\theta(n)\rangle = \langle\Omega(0)\cdot\Omega(n)\rangle = g^n. \tag{A.4}$$

Proof: The only component of interest is $\Omega_d(n+1) = \cos\theta(n+1)$; all others vanish upon averaging, by symmetry. In $d = 2$, $\cos\theta(n+1) = \cos\theta(n+1)\cos\theta_s - \sin\theta(n+1)\sin\theta_s$; the only difference for $d > 2$ is the presence of an (uncorrelated, zero-average) azimuthal factor in the second term. Therefore $\langle\cos\theta(n+1)\rangle = \langle\cos\theta(n)\rangle\langle\cos\theta_s\rangle = \langle\cos\theta(n)\rangle g$ for all n and any d. *QED*.

Equation (A.4) says that g^2 is the "effective" asymmetry factor after two scatterings, and so on. The ultimate consequence of (A.4) is that $\Omega(n)$ can be almost anywhere on Ξ_d —say, we have $\langle\cos\theta(n)\rangle \approx 1/e$— as soon as $n \approx (-\ln g)^{-1}$; if $g \lesssim 1$, this reads as

$$n \approx (1-g)^{-1}. \tag{A.5}$$

This is roughly the number of forward scatterings required for the particle to loose all memory of its original direction of travel. Finally, we note that this trend towards Ω-isotropy of multiply-scattered radiation occurs for all (non-trivial) phase functions.

A.3 Coupling Between Physical-Space and Direction-Space Statistics

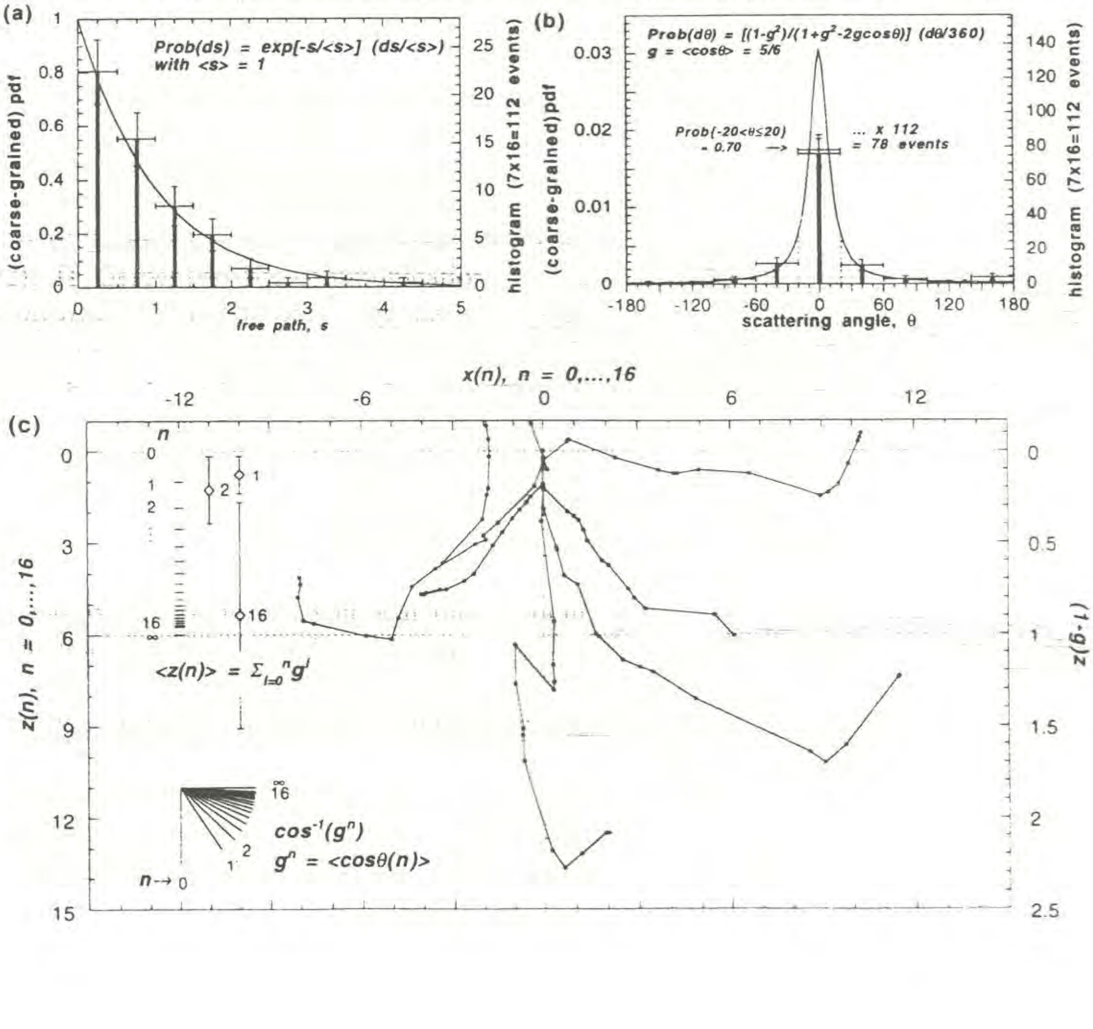
We now investigate the spatial consequences of (A.4–5). Assume the particle executes the directionally-correlated random walk, leaving the origin in direction Ω_0 at time $n = 0$; in the discrete-time picture, it is then displaced from position $x(0) = 0$ to $x(1) = s(0)\Omega(0)$, where $s(0)$ is the initial step-size. Holding $\Omega(0)$ fixed, $\langle x(1)\rangle = \langle s(0)\rangle\Omega(0) = \ell\Omega(0)$ where ℓ is the mfp. We can see from Eq. (A.1) that, after the first scattering and second step, we have $\langle x(2)\rangle = \Omega(0)[\langle s(0)\rangle + \langle s(1)\rangle\langle\Omega(0)\cdot\Omega(1)\rangle] = (1+g)\ell\Omega(0)$. In general, the assumed independence of step-sizes and step-directions leads to

$$\Omega(0)\cdot \lim_{n\to\infty}\langle x(n)\rangle = \ell\Big\langle\sum_{0}^{\infty}\cos\theta(n)\Big\rangle = \ell\sum_{0}^{\infty} g^{n} = \frac{\ell}{1-g}\,. \tag{A.6}$$

After many scatterings, the cumulative effect of forward scattering is to boost the initial ballistic motion by a factor $(1-g)^{-1}$.

In summary, an isotropic random-walker ($g = 0$) loses track of its direction of propagation at every step (average length ℓ), but it takes its Ω-correlated counterpart about $(1-g)^{-1}$ forward-biased scatterings to "forget" its original direction (cf. Fig. A1). In the process, this causes it to travel (on average) that much further. The above derivation of this well-known rescaling[15] sheds more geometrical insight than the standard route from linear transport theory to diffusion theory, using the hydrodynamic limit.[7] Furthermore, it makes *no assumption about the nature of the step-distribution*, thus generalizing the standard approach which is inherently limited to exponential step-distributions. So either normal or anomalous diffusion approximations can be used at times $\gtrsim (1-g)^{-1}$ in mean step intervals, and scales $\gtrsim$ the "transport" mfp, $\ell(1-g)^{-1}$; these quantities act as inner cut-offs to the diffusive scaling regimes.

Figure A1: *Short random walks with directional correlations in $d = 2$.* **(a)** The exponential law of extinction that dictates the distribution of free paths s: $\mathrm{Prob}\{s \geq X\} = \exp(-X/\ell)$ with unit mean ($\ell = \langle s\rangle = 1$). **(b)** The scattering kernel $p(\theta)$ in Eq. (A.3) that describes the distribution of scattering angle θ_s, with "asymmetry factor" $g = \langle\cos\theta_s\rangle = 5/6 = 0.83\cdots$. In panels (a,b), error bars are based on expected variances in number of events per bin. **(c)** Seven 2D particle trajectories starting straight down at the origin, all 16 scatterings long. The left-hand scale is in mfp's; the right uses "transport" mfp's. In the lower l.h. corner, an indication of the average direction of travel is plotted for $n = 1,\ldots,16$ and ∞; in the upper l.h. corner, the corresponding average positions are indicated for orders-of-scattering $n = 1,\ldots,16$ and ∞ (theoretical) and $n = 1$, 2, and 16 (empirical, with st. dev.'s). After a large number of anisotropic scatterings, the particle may just as well have been scattered isotropically once but the mfp for such a scattering is roughly $(1-g)^{-1} = 6$ times longer.



References

1. M.F. Shlesinger, G.M. Zaslavsky and U. Frisch, eds., *Lévy Flights and Related Topics in Physics* (Springer-Verlag, Berlin, 1995).
2. U. Frisch and H. Frisch, *Mon. Not. R. Ast. Soc.* **173**, 167 (1975).
3. T.L. Bell, U. Frisch and H. Frisch, *Phys. Rev.* A **17**, 1049 (1978).
4. H. Frisch and U. Frisch, *J. Quant. Spectros. Radiat. Trans.* **28**, 361 (1982).
5. U. Frisch and H. Frisch in Ref. 1.
6. Chandrasekhar, S., *Radiative Transfer* (Oxford University Press, Oxford, 1950)
7. K. Case and P. Zweifel, *Linear Transport Theory* (Addison–Wesley, Reading, 1967).
8. B.B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (Freeman, San Fransisco, 1977).
9. P. Lévy, *Théorie de l'Addition des Variables Aléatoires* (Gauthiers-Villars, Paris, 1937); W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 1 (Wiley, New York, 1971).
10. R.M. Goody, *Quart. J. Roy. Meteor. Soc.*, **78**, 165 (1952).
11. K.J. Falconer, *Fractal Geometry Mathematical Foundations and Applications* (Wiley, New York, 1990).
12. P. Grassberger and I. Procaccia, *Physica D* **9**, 189 (1983), *Phys Rev. Lett.* **50**, 346 (1983).
13. A. Davis, A. Marshak, W.J. Wiscombe and R.F. Cahalan in *Current Topics in Nonstationary Analysis*, eds. G. Treviño et al. (World Scientific, Singapore, 1996).
14. U. Frisch and G. Parisi in *Turbulence and Prdicability in Geophysical Fluid Dynamics and Climate Dynamics*, eds. M. Ghil, R. Benzi and G. Parisi (North–Holland, Amsterdam, 1985).
15. H.C. van de Hulst, *Multiple Light Scattering (Tables, Formulas, and Applications)*, Vol. 2 (Academic Press, New York, 1980).
16. V. Zoltarev, *One-Dimensional Stable Distributions* (Am. Math. Soc., Providence, 1986).
17. A. Davis, A. Marshak, W. Wiscombe and R. Cahalan, *J. Atmos. Sci.* **53**, 1538 (1996).
18. A. Marshak, A. Davis, W. Wiscombe and R. Cahalan, *J. Atmos. Sci.* **54** (1997, in press).
19. B. Baker, *J. Atmos. Sci.* **49**, 387 (1992); H. Gerber (private communication).
20. A. Davis, *Radiative Transfer in Scale Invariant Optical Media.* PhD thesis, Physics Department, McGill University, Montreal (1992).
21. H. Barker, *Quart. J. Roy. Meteor. Soc.* **118**, 1145 (1992).
22. E.g., A. Davis, P. Gabriel, S. Lovejoy, D. Schertzer and G.L. Austin, *J. Geophys. Res.* **95**, 11729 (1990); R.F. Cahalan, W. Ridgway, W.J. Wiscombe, T.L. Bell and J.B. Snider, *J. Atmos. Sci.* **51**, 2434 (1994); and references therein.
23. M. Tiedke, *Mon. Weath. Rev.* **124**, 745 (1995); H. Barker, *J. Atmos. Sci.* **53**, 2289 (1996).
24. A. Marshak, A. Davis, R.F. Cahalan and W.J. Wiscombe, Nonlocal Independent Pixel Approximation: Direct and Inverse Problems. *IEEE Transactions on Geoscience and Remote Sensing* (1996, submitted).
25. D. Deirmendjian, *Electromagnetic Scattering on Sphe al Polydispersions* (Elsevier, Amsterdam, 1969).
26. L.C. Henyey and J.L. Greenstein, *Astrophys. J.* **93**, 70 (1941).
27. A. Davis, P. Gabriel, S. Lovejoy and D. Schertzer in *IRS'88: Current Problems in Atmospheric Radiation*, eds. J. Lenoble and J.–F. Geleyn (Deepak, Hampton, 1989).